Origins

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What is categorification?

Informally

Def A categorification of a n-cat object C is a $(n+1)$-cat object $\tilde{C}$ s.t. $\exists$ a procedure called decategorification that recovers the original object C.

Why? $C \longrightarrow \tilde{C}$ ← more structure

learn something about C we couldn't see before

Origins: [Crane, Frenkel 1994]

nD-TQFT "invariants of n-manifolds"

2D TQFT $\longleftrightarrow$ Frob alg

3D TQFT $\longleftrightarrow$ modular $\otimes$ cat

(WRT) $\uparrow$ $U_q(\mathfrak{g})$, q root of unity

4D TQFT $\longleftrightarrow$ Cat($\mathcal{C}$)

$\longleftrightarrow$ Cat([Kuperberg 1990])

f.d Hopf alg

Ex 1: $\mathbb{Z} \rightsquigarrow$ f.d. $\mathbb{F}$ v.s.

$\dim V \xleftarrow{F} V$

or $K_0(\text{f.d. } \mathbb{F} \text{ v.s}) = \mathbb{Z}$ recover as ring!

$F(V_1 \oplus V_2) = F(V_1) + F(V_2)$

$F(V_1 \otimes V_2) = F(V_1) F(V_2)$

Ex 2: $\mathbb{Z} \rightsquigarrow \mathrm{Ch}(\mathbb{F})$

$(-1)^i \dim V_i \xleftarrow{\chi} V^\bullet$

get negatives for free!

$\chi(V_1^\bullet \oplus V_2^\bullet) = \cdots$

$\chi(V_1^\bullet \otimes V_2^\bullet) = \cdots$

Grothendieck Groups

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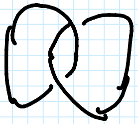
Warning: Categorifications not unique!

Ex 3 (Khovanov homology)

very computable

$L =$ 

Jones

Khovanov

$H(L) = \bigoplus_{i,j} H^{i,j}(L) \dashrightarrow \sum_{i,j} (-1)^i q^j \operatorname{rk} H^{i,j}(L) = J(L) \in \mathbb{Z}[q,q^{-1}]$

Rem: $H(L)$ is stronger invariant than $J(L)$; can detect unknot [KM]

Grothendieck Groups

Def: Let C be an additive cat. Then (split) GR

$$K_0^{\oplus}(C) = \frac{\mathbb{Z}\{[M]\}}{\langle [M] = [M_1] + [M_2] \rangle} \quad M = M_1 \oplus M_2$$

Def Let A be an abelian cat. Then

$$K_0(A) = \frac{\mathbb{Z}\{[M]\}}{[M] = [M_1] + [M_2]} \qquad 0 \to M_1 \to M \to M_2 \to 0 \text{ SES}$$

Ex: $C = B\text{-pmod}$, B f.d. alg

$$K_0^{\oplus}(B) := K_0^{\oplus}(C) = \bigoplus_{i=1}^{r} \mathbb{Z}[P_i]$$ indecomp projectives

Ex: $A = B\text{-mod}$

$$K_0(B) := K_0(A) \underset{\text{J.H.}}{=} \bigoplus_{i=1}^{r} \mathbb{Z}[L_i]$$ irr rep of B

Rem: If B ss $\Rightarrow K_0^{\oplus}(B) = K_0(B)$

Exer: $\exists$ natural bilinear form

$$\langle\ ,\ \rangle : K_0^{\oplus}(B) \times K_0(B) \to \mathbb{Z}$$

$$\langle [P], [M] \rangle = \dim \operatorname{Hom}_B(P, M)$$

Functors

Def: A functor $F: C_1 \to C_2$ is called additive if

$$F(M_1 \oplus M_2) = F(M_1) \oplus F(M_2)$$

Def A functor $F: A_1 \to A_2$ is called exact if it preserves SES

F additive $\Rightarrow [F]: K_0^{\oplus}(C_1) \to K_0^{\oplus}(C_2)$

F exact $\Rightarrow [F]: K_0(A_1) \to K_0(A_2)$

Abelian CTFN

- Throughout can replace abelian $\rightsquigarrow$ additive
  $K_0 \rightsquigarrow K_0^{\oplus}$
- Let B = k-alg with gen $\{b_i\}$ s.t.
  $b_i b_j = \sum c_{ij}^k b_k$ $\quad c_{ij}^k \in \mathbb{Z}_{\geq 0}$
- Let M be a B-mod

Def A (weak) ab CTFN of $(B, \{b_i\}, M)$ consists of an abelian cat $\tilde{M}$, iso $e: K_0(\tilde{M}) \to M$ and exact endofunctors $F_i: \tilde{M} \to \tilde{M}$ s.t.

(C1) The following diagram commutes

$$\begin{tikzcd}
K_0(\tilde{M}) \arrow[r, "{[F_i]}"] \arrow[d, "e"'] & K_0(\tilde{M}) \arrow[d, "e"] \\
M \arrow[r, "b_i"] & M
\end{tikzcd} \qquad \text{(gen)}$$

(C2) There are isomorphisms

$$F_i F_j \cong \bigoplus_k F_k^{\oplus c_{ij}^k} \qquad \text{(rel)}$$

Rem: Notice basis is fixed

Rem: We are categorifying M not B!

Ex: $B = H$, hecke alg; $\{b_i\} = \{b_w\}$ KL basis
M = reg rep of H; $\tilde{M} = SBim$, $F_w = B_w \otimes_R -$
(B_w is a graded free left R-mod)

- SBim categorifies $(H, \{b_w\}, H) \Leftrightarrow$ SCT

Relation to Khovanov Homology

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- Recall H has another basis $\{\delta_w\}_{w\in W}$

Can we categorify $\{\delta_w\}$?

- Recall $b_s = \delta_s + v \Rightarrow \delta_s = b_s - v$ bad
- Instead use $K^b(SBim)$

$$\delta_s \longrightarrow [0 \to \underline{B_s} \xrightarrow{1} R(1) \to 0] = F_s$$

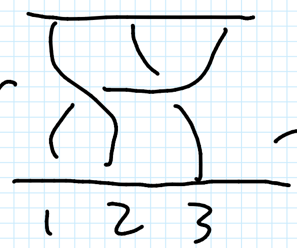
Rem: while the KL basis $\longleftarrow\!\!-\!\!-\!\!\longrightarrow$ rep theory
standard basis $\longleftarrow\!\!-\!\!-\!\!\longrightarrow$ link homology

δ_i "=" $|| \cdots \times \cdots |$ (i, $i+1$), δ_i^{-1} "=" $| \cdots \times \cdots |$ (i, $i+1$)

How to get HOMFLY homology

Given a link L, write it as $\hat{\mathcal{B}}$, $\mathcal{B}$ braid

represents element in Hecke alg



$\rightsquigarrow F_1^{-1} \otimes F_2 \in K^b(SBim)$

$W = S_n$, $s_i = (i\ \ i+1)$

- For each crossing, associate F_i or F_i^{-1}
- Take $\otimes$ of these complexes, get $F(\mathcal{B}) \in K^b(SBim)$

Decategorified	Categorified
$\mathcal{B}$	$\mathcal{B}$
$\downarrow$	$\downarrow$
$\mathcal{B} \in$ Hecke alg	$F(\mathcal{B}) \in K^b(SBim)$
$\downarrow$ Jones-Ocneanu trace	$\downarrow$ HHH
$\mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ HOMFLY-poly	triply-graded v.s HOMFLY homology
$\downarrow$ $a = q^{-1}$, $z = q^{1/2} - q^{-1/2}$	$\downarrow$??? I was lazy
$\mathbb{Z}[q^{\pm 1}]$ Jones poly	doubly-graded v.s Khovanov homology

Symmetric Functions 1

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Symmetric Functions

<u>Def</u> $Sym = \mathbb{Z}[x_1, \dots]^{S_\infty} \text{ "}\doteq\text{" } \mathbb{Z}[x_1, \dots, x_n]^{S_n}, \; n \gg 0$

aka Sym is <u>alg</u> of symmetric functions in ∞ many variables

Bases for Sym

1. monomial: $\lambda = (\lambda_1, \lambda_2, \dots)$

$$m_\lambda = \sum_{\substack{\alpha \text{ distinct} \\ \text{permutation of } \alpha}} x^\alpha$$

<u>Ex:</u> $m_{(2,1,\dots)} = x_1^2 x_2 + x_2^2 x_1 + \cdots$

2. elementary: $\lambda = (\lambda_1, \lambda_2, \dots)$

$$e_r = \sum_{i_1 < \dots < i_r} x_{i_1} \cdots x_{i_r} \; ; \; e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$$

3. complete symmetric,

$$h_n = \sum_{\lambda \vdash n} m_\lambda \; ; \; h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$$

4. power sum

$$p_n = \sum x_i^n \; ; \; p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$$

5. <u>Schur functions</u> $\lambda \vdash n$

$$s_\lambda(x_1, \dots, x_n) = \sum_{T \in SSYT(\lambda)} x^T$$

notice # of variables is finite!

<u>Ex:</u> $\lambda = (2,1)$

$s_\lambda(x_1, x_2, x_3) = x_1^2(x_2 + x_3) + x_2^2(x_1 + x_3) + x_3^2(x_1 + x_2) + 2x_1x_2x_3$

$\{p_\lambda\}$ is not a $\mathbb{Z}$-basis for Sym!

$$Sym \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[p_1, p_2, \ldots]$$

Given $\theta \in S_n$, $\theta = (\overset{q_1}{\cdots})(\overset{q_2}{\cdots})\cdots \rightsquigarrow q(\theta) = q_1 + q_2 + \cdots$

$$p_{q(\theta)} = p_{q_1} p_{q_2} \cdots$$

Ex: $\theta = (12)(34)(5) \rightsquigarrow q(\theta) = 2+2+1$

$$p_{q(\theta)} = p_2^2 p_1$$

Rem: $p_{q(\theta)} = p_{q(\tau)} \iff \theta$ is conjugate to τ in S_n

Rem: $p_{q(\theta_1 \times \theta_2)} = p_{q(\theta_1)} p_{q(\theta_2)}$

$\theta_1 \in S_n$, $\theta_2 \in S_m$ $\quad \theta_1 \times \theta_2 \in S_{n+m}$

Hall Inner Product on Sym

$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$ $\quad \langle \frac{p_\lambda}{z_\lambda}, p_\mu \rangle = \delta_{\lambda\mu}$

$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$ $\quad \langle p_n, p_m \rangle = \delta_{nm} n$

$\rightsquigarrow \langle , \rangle$ is non-deg

Thrm (Frobenius) Let $\lambda \vdash n$, $\theta \in S_n$

$$\chi_\lambda(\theta) = \langle s_\lambda, p_{q(\theta)} \rangle$$

where χ_λ = character of S^λ

S^λ = Specht module = irr rep of S_n

Categorification of Sym

- S_n "=" $\mathbb{C}[S_n]$, $\otimes = \otimes_{\mathbb{C}}$
- $S_n \times S_m \hookrightarrow S_{n+m} \rightsquigarrow \mathbb{C}[S_n] \otimes \mathbb{C}[S_m] \hookrightarrow \mathbb{C}[S_{n+m}]$

Lemma: $\mathbb{C}S_{n+m}$ is a proj $\mathbb{C}S_n \otimes \mathbb{C}S_m$ mod

Pf: Given $S_n \otimes S_m$-mod K, M

$$\mathrm{Hom}_{S_n}(K,M) \cap \mathrm{Hom}_{S_m}(K,M) = \mathrm{Hom}_{S_n \otimes S_m}(K,M)$$

Let $K = \mathbb{C}S_{n+m}$, and $0 \to M \to N \to L \to 0$ SES of $S_m \otimes S_n$-mod

kernel of $\mathrm{Hom}_{S_n \otimes S_m}(K,N) \to \mathrm{Hom}_{S_n \otimes S_m}(K,L)$ will be

$$T = \ker\left(\mathrm{Hom}_{S_n}(K,N) \cap \mathrm{Hom}_{S_m}(K,N) \to \mathrm{Hom}_{S_n}(K,L) \cap \mathrm{Hom}_{S_m}(K,L)\right)$$

$\mathbb{C}[S_n]$ is s.s $\Rightarrow$ all mod proj

$\Rightarrow T \subseteq \mathrm{Hom}_{S_n}(K,M) \cap \mathrm{Hom}_{S_m}(K,M) = \mathrm{Hom}_{S_n \otimes S_m}(K,M)$

Rem: Isn't $\mathbb{C}S_n \otimes \mathbb{C}S_m$ a s.s algebra b/c $\otimes$ of s.s. alg is s.s? Look at Rep Theory notes

$\mathrm{Rad}(A \otimes B) = \mathrm{Rad}(A) \otimes B + A \otimes \mathrm{Rad}(B)$

$\Rightarrow \mathrm{Hom}_{S_n \otimes S_m}(K,M) \subseteq T$ ⊠

Def Let Sym be category

$$\mathrm{Sym} = \bigoplus_{n \in \mathbb{N}} \mathbb{C}[S_n]\text{-mod}$$

- is an abelian cat
- is a (symmetric) monoidal cat under "induction product". Given $N \in S_n$-mod, $M \in S_m$-mod, $N \circ M \in S_{n+m}$-mod

$$N \circ M = \mathrm{Ind}_{S_n \otimes S_m}^{S_{n+m}} N \otimes_{\mathbb{C}} M$$

- $\otimes_{\mathbb{C}}$ is exact, $\otimes_{S_n \otimes S_m} S_{n+m}$ is exact by Lem

$\rightsquigarrow - \circ -$ descends to $K_0(\mathrm{Sym})$

Categorification of Sym 2

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$$K_0(Sym) = K_0\left(\bigoplus_{n\in\mathbb{N}} S_n\text{-mod}\right) = \bigoplus_{n\in\mathbb{N}} K_0(S_n\text{-mod})$$

- Each $K_0(S_n\text{-mod})$ has basis $[S^\lambda]_{\lambda\vdash n}$

$\rightsquigarrow \{[S^\lambda]\}$ is a basis for $K_0(Sym)$

Thm: There is an isomorphism of (Hopf) algs (also isometry)

$$ch: K_0(Sym) \longrightarrow Sym$$
$$[S^\lambda] \longmapsto s_\lambda$$

Pf: B/c basis is sent to basis $\Rightarrow$ ch bijection

- WTS ch is ring homomorph

Claim: $ch([V]) = \frac{1}{n!}\sum_{\sigma\in S_n} \chi_V(\sigma) P_\sigma$

Pf: $ch([S^\lambda]) = \frac{1}{n!}\sum_{\sigma\in S_n} \chi_\lambda(\sigma) P_\sigma$

$= \frac{1}{n!}\sum_{\sigma\in S_n} \langle s_\lambda, P_\sigma\rangle P_\sigma = \frac{1}{n!}\sum_{\mu\vdash n} \langle s_\lambda, p_\mu\rangle \frac{n!}{z_\mu} p_\mu$

$= \sum_{\mu\vdash n} \langle s_\lambda, \frac{p_\mu}{z_\mu}\rangle p_\mu = s_\lambda$

Theorem 4.5 (Generalized Frobenius Reciprocity). *Let A be a $\mathbb{C}-$algebra and let $H \subset G$ a finite subgroup. Let $C_G(A) = \{f : G \to A \,|\, f(xgx^{-1}) = f(g)\ \forall g, x \in G\}$ and likewise with $C_H(A)$. Define the pairing $\langle -, -\rangle_G^A : C_G(A) \times C_G(A) \to A$ by*

$$\langle f, g\rangle_G^A = \frac{1}{|G|}\sum_{x\in G} f(x)g(x^{-1})$$

Now given $\psi \in C_H(A)$ and $\chi \in C_G(A)$ we have that

$$\langle \mathrm{Ind}_H^G(\psi), \chi\rangle_G^A = \langle \psi, \chi|_H\rangle_G^A$$

where $\mathrm{Ind}_H^G(\psi) = \frac{1}{|H|}\sum_{x\in G} \psi(x^{-1}gx)$, $\psi(a) = 0$ if $a \notin H$.

$A = Sym$, $G = S_n$, $\psi(\sigma) := P_{q(\sigma)} \in C_{S_n}(Sym)$

$\Rightarrow ch([V]) = \langle \chi_V, \psi(\sigma)\rangle$

$\Rightarrow ch([N]\circ[M]) = ch\left(\left[\mathrm{Ind}_{S_n\times S_m}^{S_{n+m}} N\otimes M\right]\right)$

$= \left\langle \mathrm{Ind}_{S_n\times S_m}^{S_{n+m}} N\otimes M, \psi(\sigma)\right\rangle$

$= \langle \chi_N \chi_M, \psi|_{S_n\times S_m}\rangle$

$= \langle \chi_N\chi_M, \psi_{S_n}\psi_{S_m}\rangle \quad (P_{q(\sigma_1\times\sigma_2)} = P_{q(\sigma_1)}P_{q(\sigma_2)})$

$= \langle \chi_N, \psi_{S_n}\rangle\langle \chi_M, \psi_{S_m}\rangle = ch([N])ch([M])$

$$= \langle \chi_n, \psi_{s_n} \rangle \langle \chi_m, \psi_{s_m} \rangle = ch([N])ch([M])$$

Heisenberg 1

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Heisenberg lie alg

Def (rk 1) $\mathfrak{h}_{\mathbb{Q}}$ is lie alg w/ gen $\{b_n\}_{n\in\mathbb{Z}}$

$$[b_n, b_m] = n\,\delta_{n,-m}$$

$\mathfrak{h}_{\mathbb{Q}}$ "=" $U(\mathfrak{h}_{\mathbb{Q}})$

Def (Fock space) Let $\mathfrak{h}_{\mathbb{Q}}^{+} \subseteq \mathfrak{h}_{\mathbb{Q}}$ be subalg gen by $\{b_n\}_{n\geq 0}$. $\mathfrak{h}_{\mathbb{Q}}^{+}$ is commutative, so has 1-dim trivial rep $\mathbb{C}$.

$$\pi := \mathrm{Ind}_{\mathfrak{h}_{\mathbb{Q}}^{+}}^{\mathfrak{h}_{\mathbb{Q}}} \mathbb{C} \;\underline{\underline{\text{Set}}}\; \mathbb{C}[b_{-i}]_{i\in\mathbb{Z}^{+}}$$

$$\text{"="}\ \mathbb{C}[p_1, p_2, \dots]$$

Explicitly, the action of $\mathfrak{h}_{\mathbb{Q}}$ is

- mult by b_n, $n<0$
- $b_n = n\frac{\partial}{\partial b_{-n}}$, $n \geq 0$

Notice if we let $\{p_i\} \curvearrowright \mathrm{Sym}$ by left mult $\{p_i\}$ mimics action of $\{b_n\}_{n<0} \curvearrowright \pi$

Q: Can we also mimic action of $\{b_n\}_{n\geq 0}$?

– Recall $\langle , \rangle$ on Sym is non-deg. As mult by f is linear $\exists$ lin op f^*

$$\langle f^*(a), b\rangle = \langle a, f(b)\rangle \quad \forall a,b\in \mathrm{Sym}$$

$$p_i^*(p_n) = \sum \langle p_i^*(p_n), \tfrac{p_\lambda}{z_\lambda}\rangle p_\lambda$$

$$= \sum \langle p_n, \tfrac{p_i p_\lambda}{z_\lambda}\rangle p_\lambda \overset{\lambda=0}{=} \langle p_n, p_i\rangle$$

$$= i\,\delta_{in}$$

Heisenberg 2

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$p_i^*(p_n) = i\delta_{in}$ matches $b_i(b_{-n}) = i\frac{\partial}{\partial b_{-i}}(b_{-n})$, $i>0$, $n>0$

Lemma: p_i^* acts by $i\frac{\partial}{\partial p_i}$ on $\mathrm{Sym}_{\mathbb{Q}} = \mathbb{Q}[p_1, \dots, p_n]$ and thus gives an action of $\mathfrak{h}_{\mathbb{Q}} \circlearrowright \mathrm{Sym}_{\mathbb{Q}} \cong \pi$

Problem: For purposes of cat, want to work w/ Sym! Need integral version of $\mathfrak{h}_{\mathbb{Q}}$!

Thm π is a faithful rep of $\mathfrak{h}_{\mathbb{Q}}$

$\rightsquigarrow \mathfrak{h}_{\mathbb{Q}} \hookrightarrow \mathrm{End}_{\mathbb{C}}\pi \cong \mathrm{End}_{\mathbb{C}}\mathrm{Sym}_{\mathbb{Q}}$

Idea: Replace $\mathrm{Sym}_{\mathbb{Q}}$ w/ $\mathrm{Sym} = \mathbb{Z}[e_1, e_2, \dots] = \mathbb{Z}[h_1, \dots]$

and have $\mathfrak{h}$ act via mult by e_n and h_n^*

Def $\mathfrak{h}_{\mathbb{Z}}$ is the $\mathbb{Z}$-subalg of $\mathrm{End}_{\mathbb{Z}}\mathrm{Sym}$ gen by mult by $f \in \mathrm{Sym}$ and the linear operators f^*

Rem: $\mathfrak{h}_{\mathbb{Z}}$ has gen $\{e_n\}_{n\geq 1}, \{h_n^*\}_{n\geq 1}$ w/ relations

- $h_m^* e_n = e_n h_m^* + e_{n-1} h_{m-1}^*$
- $e_m e_n = e_n e_m$
- $h_m^* h_n^* = h_n^* h_m^*$

Categorification of Fock Space
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Cat of Fock space

- Want weak ab cat of $(\mathfrak{h}_{\mathbb{Z}}, \{e_n, h_n^*\}, \mathrm{Sym})$
- Clearly $\widetilde{\mathrm{Sym}} = \mathcal{S}ym = \bigoplus_{n\in\mathbb{N}} S_n\text{-mod}$, $\varphi = ch$
- Need to define functors: F_{e_n}, $F_{h_n^*}$

Def: Suppose $K \in S_\ell \otimes S_m\text{-mod}$, $M \in S_m\text{-mod}$

K is a S_ℓ, S_m-mod via $S_\ell \to S_\ell \otimes S_m$, etc.

Then S_ℓ-mod structure on $\mathrm{Hom}_{S_m}(M, K)$ is

$$(s\cdot f)(m) = s\cdot f(m)$$

Def For each $M \in S_m\text{-mod}$, $N \in S_n\text{-mod}$

$$\mathrm{Ind}_M(N) = \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} M \otimes N \in S_{m+n}\text{-mod}$$

$$\mathrm{Res}_M(N) = \mathrm{Hom}_{S_m}(M, \mathrm{Res}_{S_{n-m}\times S_m}^{S_n}(N)) \in S_{n-m}\text{-mod}$$

Note: - We define $\mathrm{Res}_{S_m}(N) = 0$ if $n-m<0$

- $\mathrm{Ind}_M(N) = M \circ N$
- Both functors are exact, so get induced maps $[\mathrm{Ind}_M] : G_0(\mathcal{S}ym) \to G_0(\mathcal{S}ym)$

Lemma: $\mathrm{Ind}_M \dashv \mathrm{Res}_M$

Lemma 4.6 (Slightly Different Tensor-Hom). *Let L be a $S-$module, M be a $R-$module and N be a $S \otimes_{\mathbb{C}} R$ module. Then N will be a S and R module via the algebra homomorphisms $S \to S \otimes_{\mathbb{C}} R$ and $R \to S \otimes_{\mathbb{C}} R$. Give* $\mathrm{Hom}_R(M, N)$ *the structure of a $S-$module by post-acting, i.e. $(s \cdot f)(m) = s \cdot f(m)$. Then we have a natural isomorphism*

$$\mathrm{Hom}_{S\otimes R}(L \otimes_{\mathbb{C}} M, N) \cong \mathrm{Hom}_S(L, \mathrm{Hom}_R(M, N))$$

Pf: Let $M \in S_m\text{-mod}$, $N \in S_n\text{-mod}$, $L \in S_{n-m}\text{-mod}$

$$\mathrm{Hom}_{S_n}(\mathrm{Ind}_M(L), N) = \mathrm{Hom}_{S_n}(\mathrm{Ind}_{S_{n-m}\times S_m}^{S_n}(L \otimes M), N)$$

$$\overset{\text{Frob}}{=} \mathrm{Hom}_{S_{n-m}\times S_m}(L \otimes M, \mathrm{Res}_{S_{n-m}\times S_m}^{S_n}(N))$$

$$\overset{\text{lem}}{=} \mathrm{Hom}_{S_{n-m}}(L, \mathrm{Hom}_{S_m}(M, \mathrm{Res}_{S_{n-m}}^{S_n}(N)))$$

Prop: Let $F_{e_n} = \mathrm{Ind}_{S^{(1,\dots,1)}}$, $F_{h_n^*} = \mathrm{Res}_{S^n}$. Then (C1) is satisfied.

$$\begin{tikzcd} K_0(\mathcal{S}ym) \arrow[r, "{[\mathrm{Ind}_m]}"] \arrow[d, "ch"'] & K_0(\mathcal{S}ym) \arrow[d, "ch"] \\ \mathrm{Sym} \arrow[r, "{ch[m]\times -}"'] & \mathrm{Sym} \end{tikzcd} \qquad \begin{tikzcd} K_0(\mathcal{S}ym) \arrow[r, "{[\mathrm{Res}_m]}"] \arrow[d, "ch"'] & K_0(\mathcal{S}ym) \arrow[d, "ch"] \\ \mathrm{Sym} \arrow[r, "{ch([m])^*}"'] & \mathrm{Sym} \end{tikzcd}$$

Pf: $ch \circ [\mathrm{Ind}_m](N) = ch(\mathrm{Ind}_m(N)) = ch(M \circ N)$

$\overset{\text{ring}}{=} ch M \, ch N = ch M \times (ch(N))$ $(*)$

$\langle ch L, ch(\mathrm{Res}_m(N))\rangle \overset{\text{isom}}{=} \langle L, \mathrm{Res}_m(N)\rangle \overset{\text{def}}{=}$

$\dim \mathrm{Hom}_{S_{n-m}}(L, \mathrm{Res}_m(N)) \overset{\text{adj}}{=} \dim \mathrm{Hom}_{S_n}(\mathrm{Ind}_m(L), N) =$

$\langle \mathrm{Ind}_m(L), N\rangle = \langle M \circ L, N\rangle = \langle ch\, M\circ L, ch N\rangle$

$\overset{(*)}{=} \langle ch M \times ch L, ch N\rangle$

$= \langle ch L, ch\, M^*(ch N)\rangle$

- ch is iso, $\langle , \rangle$ non-deg $\Rightarrow$ $ch(\mathrm{Res}_m(N)) = ch\, M^*(ch(N))$

- To complete weak cat, need to check rel (C2)
- Here it's very important what F_{e_n}, $F_{h_n^*}$ are

Prop (Weak Cat of Fock Space) We have following isomorphisms of functors

- $F_{e_m} \circ F_{e_n} \simeq F_{e_n} \circ F_{e_m}$
- $F_{h_m^*} \circ F_{h_n^*} \simeq F_{h_n^*} \circ F_{h_m^*}$
- $F_{h_m^*} \circ F_{e_n} \simeq F_{e_n} \circ F_{h_m^*} \oplus F_{e_{n-1}} \circ F_{h_{m-1}^*}$

Tools for pf: (1)

Lemma 4.7. *Let $H \subset G$ be a subgroup of G and let N be a $H-$module and let M be an R module. Then we have a natural isomorphism of $R \otimes G-$modules*

$$M \otimes_{\mathbb{C}} \mathrm{Ind}_H^G(N) \cong \mathrm{Ind}_{R\otimes H}^{R\otimes G}(M \otimes N)$$

(2) Mackey iso:

Proposition 7
Let (W, ρ) be a representation of $H \subset G$. For $s \in K \setminus G/H$ let $H_s = sHs^{-1} \cap K$ a subgroup of K. Let (ρ^s, W_s) be the representation of H_s given by $\rho^s(x) = \rho(s^{-1}xs)$ for $x \in H_s$. We will then have

$$\mathrm{Res}_K^G\left(\mathrm{Ind}_H^G(W)\right) \cong \bigoplus_{s \in K\setminus G/H} \mathrm{Ind}_{H_s}^K W_s$$

as s ranges over a set of representatives for the double coset $K \setminus G/H$.

Base Case: $m = n = 1 \Rightarrow h_1 = e_1 = P_1$, $S^1 = S^1 =$ trivial rep of $\mathbb{C}[S_1] = \mathbb{C}$

- $F_{h_1^*}(N) = \mathrm{Hom}_{S_1}(\mathrm{triv}, \mathrm{Res}^{S_n}_{S_{n-1}\times S_1}(N)) = \mathrm{Res}^{S_n}_{S_{n-1}}(N)$
- $F_{e_1}(N) = \mathrm{Ind}^{S_{n+1}}_{S_1 \times S_n}(\mathrm{triv} \otimes N) = \mathrm{Ind}^{S_{n+1}}_{S_n}(N)$

Our relation becomes

$$\mathrm{Res}^{S_{n+1}}_{S_n} \mathrm{Ind}^{S_{n+1}}_{S_n} (N) = \mathrm{Ind}^{S_n}_{S_{n-1}} \mathrm{Res}^{S_n}_{S_{n-1}} (N) \oplus \mathrm{Id}(N) \quad (p_1^* p_1 - p_1 p_1^* = 1)$$

- You can use Mackey Iso to show this : $S_n \backslash S_{n+1} / S_n$ only has 2 double cosets
- Or use branching rules for irreducibles